Digit Sums

Jonathan Frech

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Throughout this document the notation $0 \in \mathbb{N} \neq \mathbb{N}^+$ is used. Furthermore, the letter $b \geq 2$ will always denote a base.

Theorem (Unique b-adic representation)**.** *Let $n \in \mathbb{N}$ a natural number. There exists a unique b-adic representation*

$$n = \sum_{j=0}^{k} a_j \cdot b^j$$

where $k \in \mathbb{N}$ is minimally chosen with $0 \leq a_j < b$ for all $0 \leq j \leq k$.

Proof. Existence of the above representation will first be shown by inducting on natural numbers less than b^{k+1}.
Base case. For natural numbers $0 \leq n < b^1$ the representation $a_0 := n$ exists.
Induction step. Let $0 \leq n < b^{k+1}$ be a natural number. Euclidian division gives natural numbers $0 \leq n_\star < b^k$ and $0 \leq a_\star < b$ such that $n = b \cdot n_\star + a_\star$. By induction on $n_\star$ the representation

$$n = b \cdot n_\star + a_\star = b \cdot \left(\sum_{j=0}^{k_\star} a_j \cdot b^j \right) + a_\star$$

exists.

To show uniqueness, note that for all $k \in \mathbb{N}$ and $0 \leq a_j < b$ the inequality

$$b^k > b^k - 1 = (b-1) \cdot \sum_{j=0}^{k-1} b^j \geq \sum_{j=0}^{k-1} a_j \cdot b^j$$

holds. Now suppose, a number had two different representations. Since both contain finitely many digits, there has to be a maximum index j such that the digits $a_j \neq a'_j$ differ. However, since both represent the same and by the above inequality this requires two digits with a higher index to differ, contradicting the maximality of j. □

Definition (Digit sum). *Let $n \in \mathbb{N}$ a natural number with its unique b-adic representation $n = \sum a_j b^j$. Let*

$$\mathrm{ds}_b(n) := \sum_{j=0}^{k} a_j$$

denote the b-adic digit sum of n.

Note that $\mathrm{ds}_b\big|_{\mathbb{N}_{<b}} = \mathrm{id}\big|_{\mathbb{N}_{<b}}$ as well as that $\mathrm{ds}_b\big|_{\mathbb{N}_{\geq b}}$ is strictly monotonically decreasing.

Definition (Finite fixpoint limit). *Let A be a set and $f : A \to A$ a mapping. It denotes*

$$f^\infty : A \to A, x \mapsto \begin{cases} x & \text{if } x = f(x), \\ f^\infty(f(x)) & \text{otherwise} \end{cases}$$

the finite fixpoint limit of f, provided the recursion terminates.

Remark (Continuation of transitive relations). *Let A be a set, $f : A \to A$ a mapping for which f^∞ exists and $\sim \subset A \times A$ a transitive relation such that $x \sim f(x)$ for all $x \in A$. Then*

$$x \sim f^\infty(x)$$

follows for all $x \in A$.

Proof. Since f^∞ exists, the chain

$$x \sim f(x) \sim f(f(x)) \sim \cdots \sim f^\infty(x)$$

requires only finitely many relations, implying per transitivity $x \sim f^\infty(x)$. □

Definition (Final digit sum). *Let*

$$\mathrm{fds}_b := \mathrm{ds}_b^\infty$$

denote the b-adic final digit sum, whereby its existence is guaranteed by $\mathbb{N}$ being well-ordered and by ds_b strictly monotonically decreasing for non-fixpoints.

Note that $\mathrm{im}(\mathrm{fds}_b) = \mathbb{N}_{<b}$.

Theorem (Congruence to the digit sum)**.** *Let $n \in \mathbb{N}$ be a natural number. Then*

$$n \equiv \mathrm{ds}_b(n) \mod (b-1).$$

Proof.

$$\begin{aligned} n &= \sum_{j=0}^{k} a_j \cdot b^i = \sum_{j=0}^{k} a_j + \sum_{j=1}^{k} a_j \cdot (b^j - 1) \\ &= \mathrm{ds}_b(n) + \sum_{j=1}^{k} a_j \cdot \left((b-1) \cdot \sum_{i=0}^{j-1} b^i \right) \\ &= \mathrm{ds}_b(n) + (b-1) \cdot \left(\sum_{j=1}^{k} a_j \cdot \sum_{i=0}^{j-1} b^i \right) \\ &\equiv \mathrm{ds}_b(n) \mod (b-1) \end{aligned}$$

□

Lemma (Congruence implies divisor equivalence)**.** *Let $m, n, n', d \in N$ be natural numbers such that*

$$n \equiv n' \mod m$$

as well as $d \mid m$. Then

$$d \mid n \iff d \mid n'.$$

Proof. With $d \mid m$ and $m \mid (n - n')$ it follows that

$$d \mid (n - n')$$

and thus divisor equivalence.

Note that the reverse does not hold, as $\{m = 6, n = 5, n' = 7\}$ shows. □

Corollary (Divisibility equivalence)**.** *Let $n, d \in \mathbb{N}$ be two natural numbers with $d \mid (b-1)$. Then*

$$d \mid n \iff d \mid \mathrm{ds}_b(n).$$

Proof. Combining both *Congruence to the digit sum* and *Congruence implies divisor equivalence.* □

Remark (A family of constant final digit sums)**.** *The final digit sum in base ten of three consecutive positive natural numbers of which the largest is divisible by three is six.*[1]

Proof. Let $n \in \mathbb{N}^+$ be a positive natural number. One calculates

$$\begin{aligned} \mathrm{fds}_{10}\left(\sum_{j=0}^{2}(3n-j)\right) &= \mathrm{fds}_{10}(3n-2+3n-1+3n) \\ &= \mathrm{fds}_{10}(9n-3) \\ &\overset{\star}{\equiv} 9n-3 \mod 9 \\ &\equiv 6 \mod 9, \end{aligned}$$

whereby $\star$ follows from both *Congruence to the digit sum* and *Continuation of transitive relations*. Since $\mathrm{im}(\mathrm{fds}_{10}) = \mathbb{N}_{<10}$, equality follows. ☐

[1]Factoid taken from Child, Lee: *Der Anhalter*. München: Blanvalet, 2015; p. 73.